

# THE MORSE COMPLEX FOR A MORSE FUNCTION ON A MANIFOLD WITH CORNERS

DAVID G.C. HANDRON

ABSTRACT. A Morse function  $f$  on a manifold with corners  $M$  allows the characterization of the Morse data for a critical point by the Morse index. In fact, a modified gradient flow allows a proof of the Morse theorems in a manner similar to that of classical Morse theory. It follows that  $M$  is homotopy equivalent to a CW-complex with one cell of dimension  $\lambda$  for each essential critical point of index  $\lambda$ . The goal of this article is to determine the boundary maps of this CW-complex, in the case where  $M$  is compact and orientable.

First, the boundary maps are defined in terms of the modified gradient flow. Then a transversality condition is imposed which insures that the attaching map is non-degenerate in a neighborhood of each critical point. The degree of this map is then interpreted as a sum of trajectories connecting two critical points each counted with a multiplicity determined by a choice of orientations on the tangent spaces of the unstable manifold at each critical point.

## 1. INTRODUCTION

Goresky and MacPherson's *Stratified Morse Theory* ([GM]) represented a great step forward in extending the ideas of Morse, Thom and Smale to spaces more general than smooth manifolds. The complexities involved in dealing with stratified spaces prompted Goresky and MacPherson to comment on the nostalgia their Stratified Morse Theory might inspire for the classical version, where the Morse data of a critical point is determined by a single number, the Morse index.

In [Va] Vakhrameev proved the Morse theorems in the setting of manifolds with corners, a setting which strikes a nice balance between the simplicity of classical Morse theory on one hand, and the generality of Stratified Morse Theory on the other. Manifolds with corners are a class of stratified space that arise naturally in many applications, yet on these spaces the Morse data for an (essential) critical point still is determined by a single number.

Others have studied Morse Theory in similar settings, (e.g. Hamm ([Ha1], [Ha2]), Siersma ([Si]) and Braess ([Br]), but none of these make use of a global flow on manifolds with corners. The Morse theory for manifolds with corners developed in [Han] utilizes a modified gradient flow to prove the theorems in a more classical manner (e.g. as in [Mi]). This allows, as we shall see, the construction of a Morse complex with boundary maps determined by the trajectories connecting (essential) critical points.

---

*Date:* February 1, 2008.

*1991 Mathematics Subject Classification.* 57R70, 57R25.

*Key words and phrases.* Morse theory, stratified spaces, manifolds with corners.

**1.1. Setup and Definitions.** Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_\ell\}$  denote the standard basis vectors in  $\mathbb{R}^n$ . Define  $\mathbb{H}_\ell^n$  to be the set

$$\mathbb{H}_\ell^n = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} \cdot \mathbf{e}_i \geq 0 \text{ for all } i > n - \ell\},$$

where  $\cdot$  denotes the standard inner product on  $\mathbb{R}^n$ .

**Definition 1.** An  $n$ -dimensional manifold with corners is a topological space  $M$  together with an atlas  $\mathcal{A}$  of charts  $\mathbf{x}_a : U_a \rightarrow \mathbb{H}_{\ell_a}^n$  such that  $\{U_a\}_a$  is an open cover of  $M$ . We say that  $(M, \mathcal{A})$  is a  $C^r$ -manifold with corners if each transition function  $\mathbf{x}_b \mathbf{x}_a^{-1} : \mathbf{x}_a(U_a \cap U_b) \rightarrow \mathbf{x}_b(U_a \cap U_b)$  is of class  $C^r$ , i.e. can be extended to a  $C^r$  map between open sets of  $\mathbb{R}^n$ .

The tangent space  $T_p M$  of a manifold with corners  $M$  at a point  $p$  may be thought of as equivalence classes of the set

$$\{(\mathbf{x}, \mathbf{v}) : \mathbf{x} \text{ is a coordinate chart at } p \in M \text{ and } \mathbf{v} \in \mathbb{R}^n\},$$

where  $(\mathbf{x}, \mathbf{v}) \sim (\mathbf{y}, \mathbf{w})$  if  $D(\mathbf{x} \circ \mathbf{y}^{-1})(\mathbf{w}) = \mathbf{v}$ . Then for each  $p \in M$ ,  $T_p M$  is an  $n$ -dimensional vector space.

If  $p \in M$ , we will say a coordinate chart *at*  $p$  is a chart  $\mathbf{x}_p \in \mathcal{A}$  such that  $\mathbf{x}_p(p) = \mathbf{0} \in \mathbb{H}_\ell^n$ . In this case, the number  $\ell = \ell(p)$  is uniquely determined by  $p$ . This allows consideration of  $M$  as a stratified space. Each connected component of  $\mathcal{E}_\ell = \{p \in M : \ell(p) = \ell\}$  is a stratum of dimension  $n - \ell$ .

**Definition 2.** A tangent vector in  $T_p M$  points *outward* if some representative  $(\mathbf{x}, \mathbf{v})$  has  $\mathbf{v} \notin \mathbb{H}_{\ell(p)}^n$ . A tangent vector in  $T_p M$  points *inward* (or *into*  $M$ ) if some representative  $(\mathbf{x}, \mathbf{v})$  has  $\mathbf{v} \in \mathbb{H}_{\ell(p)}^n$ .

These terms are well defined, since for any two coordinate charts at  $p$ , the transition functions preserve  $\mathbb{H}_{\ell(p)}^n$ . Note that the definition of an inward pointing vector includes those vectors which are tangent to the boundary of  $M$ .

If  $K$  is a stratum of  $M$  then  $K$  is a manifold. We may thus consider at a point  $p \in K$  the tangent space  $T_p K$ . In fact,  $T_p K \subseteq T_p M$  as

$$T_p K = \{\mathbf{v} \in T_p M \mid \mathbf{v} \text{ and } -\mathbf{v} \text{ point inward}\}$$

If  $p \in K$  is in the closure of another stratum,  $L$ , we can define the generalized tangent space

$$T_p L = \{\mathbf{w} \in T_p M : \mathbf{w} = \lim_{i \rightarrow \infty} \mathbf{v}_i \in T_{q_i} K \text{ for some sequence } \{q_i\} \rightarrow p\}.$$

We may also write this as  $T_p L = \lim_{q \rightarrow p} T_q L$ .

**Definition 3.** A *Riemannian metric* on a  $C^r$ -manifold with corners  $M$  consists of an inner product  $\langle \cdot, \cdot \rangle_p$  on each tangent space  $T_p M$ , such that with respect to any coordinate chart  $\mathbf{x} : U \rightarrow \mathbb{H}_\ell^n$ , the functions  $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle : \mathbf{x}(U) \rightarrow \mathbb{R}$  are of class  $C^r$ , i.e. can be extended to a  $C^r$  function on an open set in  $\mathbb{R}^n$ .

We say that a coordinate chart  $\mathbf{x} : U \rightarrow \mathbb{H}_\ell^n$  at a point  $p$  is a *standard* coordinate chart if whenever  $i \leq n - \ell$  and  $j > n - \ell$  the tangent vectors  $\frac{\partial}{\partial x_i}$  and  $\frac{\partial}{\partial x_j}$  are orthogonal.

**Lemma 1.** For each point  $p \in M$ , there is a standard coordinate chart at  $p$ .

*Proof.* Let  $\mathbf{y} : U \rightarrow \mathbb{H}_\ell^n \cong \mathbb{R}^{n-\ell} \times [0, \infty)^\ell$  be a coordinate chart at  $p$ . Then  $\mathbf{y}(p) = \mathbf{0}$ . Define an inner product on  $\mathbb{R}^n$  by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{y}_*^{-1}(\mathbf{v}), \mathbf{y}_*^{-1}(\mathbf{w}) \rangle_p.$$

Here we have used the natural identification of  $\mathbb{R}^n$  with  $T_{\mathbf{0}}\mathbb{R}^n$ .

Using this inner product, define an orthogonal projection onto  $\mathbb{R}^{n-\ell}$ . For each vector  $\mathbf{v} \in \mathbb{R}^n$ ,  $\text{proj}(\mathbf{v}) \in \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_{n-\ell})$  is the unique vector such that  $\langle \mathbf{v} - \text{proj}(\mathbf{v}), \mathbf{w} \rangle = 0$  for every  $\mathbf{w} \in \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_{n-\ell})$ .

Now define a new basis for  $\mathbb{R}^n$  by

$$\begin{aligned} \bar{\mathbf{e}}_i &= \mathbf{e}_i & \text{for } i \leq n - \ell \\ \bar{\mathbf{e}}_j &= \mathbf{e}_j - \text{proj}(\mathbf{e}_j) & j > n - \ell. \end{aligned}$$

Then if  $i \leq n - \ell$  and  $j > n - \ell$ ,  $\langle \bar{\mathbf{e}}_i, \bar{\mathbf{e}}_j \rangle = 0$ .

Let  $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the linear transformation that carries each  $\mathbf{e}_k$  to  $\bar{\mathbf{e}}_k$ . Then  $J$  is an isomorphism. We wish to show that  $J : \mathbb{H}_\ell^n \rightarrow \mathbb{H}_\ell^n$ . It is sufficient to show that if

$$\mathbf{v} = a_1 \mathbf{e}_1 + \dots + a_n \mathbf{e}_n$$

with  $a_j \geq 0$  whenever  $j > n - \ell$ , then the expression

$$\mathbf{v} = \bar{a}_1 \bar{\mathbf{e}}_1 + \dots + \bar{a}_n \bar{\mathbf{e}}_n$$

satisfies  $\bar{a}_j \geq 0$  whenever  $j > n - \ell$ . But if  $\mathbf{v} = \bar{a}_1 \bar{\mathbf{e}}_1 + \dots + \bar{a}_n \bar{\mathbf{e}}_n$ , then

$$\begin{aligned} \mathbf{v} &= \bar{a}_1 \mathbf{e}_1 + \dots + \bar{a}_{n-\ell} \mathbf{e}_{n-\ell} \\ &\quad + \bar{a}_{n-\ell+1} (\mathbf{e}_{n-\ell+1} - \text{proj}(\mathbf{e}_{n-\ell+1})) + \dots + \bar{a}_n (\mathbf{e}_n - \text{proj}(\mathbf{e}_n)). \end{aligned}$$

Then, setting  $\text{proj}(\mathbf{e}_j) = \sum_{k=1}^{n-\ell} p_{j,k} \mathbf{e}_k$ , we see that

$$\begin{aligned} \mathbf{v} &= \left( \bar{a}_1 - \sum_{j=n-\ell+1}^n p_{j,1} \right) \mathbf{e}_1 + \dots + \left( \bar{a}_{n-\ell} - \sum_{j=n-\ell+1}^n p_{j,n-\ell} \right) \mathbf{e}_{n-\ell} \\ &\quad + \bar{a}_{n-\ell+1} \mathbf{e}_{n-\ell+1} + \dots + \bar{a}_n \mathbf{e}_n. \end{aligned}$$

So for  $j > n - \ell$ ,  $\bar{a}_j = a_j$ . It follows that  $J : \mathbb{H}_\ell^n \rightarrow \mathbb{H}_\ell^n$ .

Finally, let  $\mathbf{x} = J \circ \mathbf{y}$ . Then  $\mathbf{x} : U \rightarrow \mathbb{H}_\ell^n$  and  $\mathbf{x}(p) = \mathbf{0}$ , so  $\mathbf{x}$  is a standard coordinate chart at  $p$ . Moreover,

$$\mathbf{x}_*^{-1} \left( \frac{\partial}{\partial x_k} \right) = J \circ \mathbf{y}_*^{-1} \left( \frac{\partial}{\partial y_k} \right).$$

Since

$$\mathbf{y}_*^{-1} \left( \frac{\partial}{\partial y_k} \right) = \mathbf{e}_k,$$

it follows that  $\mathbf{x}_*^{-1} \left( \frac{\partial}{\partial x_k} \right) = \bar{\mathbf{e}}_k$ , and  $\mathbf{x}$  is a standard coordinate chart at  $p$ .

q.e.d.

Let  $M$  be a compact,  $n$ -dimensional  $C^r$ -manifold with corners ( $r \geq 2$ ) endowed with a Riemannian metric. Let  $f : M \rightarrow \mathbb{R}$  be a  $C^r$  function. For every stratum  $K$  of  $M$ , the function  $f|_K : K \rightarrow \mathbb{R}$  is a  $C^r$  function on the manifold  $K$ . If  $p \in K$ , we say  $p$  is a *critical point* of  $f$  whenever  $p$  is a critical point of  $f|_K$ . Recall that a critical point of  $f|_K$  is *non-degenerate* if the Hessian of  $f|_K : K \rightarrow \mathbb{R}$  is non-zero at  $p$ .

If  $p$  is an interior point of  $M$ , then it is contained in a stratum  $K$  with dimension  $n$ . The gradient  $\nabla f$  of the function  $f : M \rightarrow \mathbb{R}$  has a natural definition.  $K$  is a manifold with a Riemannian metric induced by the metric on  $M$ . We let

$\nabla f(p) = \nabla f|_K(p) \in T_p K = T_p M$ . If  $p$  is in the boundary of  $M$ , we can define  $\nabla f(p) = \lim_{q \rightarrow p} \nabla f(q)$  where  $q$  is taken to be in the interior of  $M$ .

We can now define the class of functions we wish to consider:

**Definition 4.** Let  $f : M \rightarrow \mathbb{R}$  be a  $C^r$  function from a compact,  $n$ -dimensional  $C^r$ -manifold with corners ( $r \geq 2$ ) endowed with a Riemannian metric into the real numbers. Then  $f$  is a Morse function on if the following hold:

- (1) If  $K$  is a stratum of  $M$ , and  $p \in K$  is a critical point of  $f|_K : K \rightarrow \mathbb{R}$ , then either
  - (a)  $p$  is a non-degenerate critical point of  $f|_K : K \rightarrow \mathbb{R}$  or
  - (b) the vector  $-\nabla f(p)$  points into  $M$ .
- (2) If  $p \in K$  is a critical point of  $f|_K$ , then for any stratum  $L$  with  $p$  in the closure of  $L$ ,  $-\nabla f(p) \notin T_p L^\perp \subseteq T_p M$ .
- (3) If  $-\nabla f$  is tangent to a stratum  $K$  at a point  $p \in K$ , and  $\mathbf{x} : U \rightarrow \mathbb{H}_\ell^n$  is a standard coordinate chart at  $p$ , then for  $j > n - \ell$ , the directional derivative of  $\frac{\partial f}{\partial x_j}$  in the direction  $-\nabla f(p)$  is non-zero.

This definition is inspired by the definition of Morse function in [GM]. It differs in condition (1b), where the critical point is not required to be non-degenerate, and in the addition of condition (3), which ensures that the vector field  $G$  defined below produces a continuous flow.

We say that  $p$  is an *essential critical point* of  $f$  if  $f$  satisfies condition (1a) at  $p$ , but not condition (1b). The *index* of an essential critical point  $p$  in a stratum  $K$  is equal to its index (in the classical sense) as a critical point of  $f|_K$ .

Suppose  $f : M \rightarrow \mathbb{R}$  is a Morse function such that  $f^{-1}(-\infty, r]$  is compact for each  $r \in \mathbb{R}$ . It is shown in [Han] that  $M$  is homotopy equivalent to a CW-complex with one cell of dimension  $\lambda$  for each essential critical point of index  $\lambda$ .

Let  $G$  be the *modified gradient vector field* of  $f$  on  $M$ , defined in [Han] by projecting the vector  $-\nabla f(p)$  onto the stratum of highest dimension such that the resulting vector does not point outward from  $M$ . It is shown in [Han] that  $G$  induces a flow  $\varphi : [0, \infty) \times M \rightarrow M$  and that the stationary points of this flow are exactly the essential critical points of  $f$ . The material above can be found in greatly expanded form in [Han].

We can define the stable and unstable sets of an essential critical point  $p$  by

$$S(p) = \left\{ q \in M : \lim_{t \rightarrow \infty} \varphi(t, q) = p \right\},$$

and

$$U(p) = \left\{ q \in M : \exists \{q_k\}_{k=1}^\infty \subset M \text{ such that } \varphi(k, q_k) = q \text{ and } \lim_{k \rightarrow \infty} q_k = p \right\}.$$

## 2. RESULTS PROVED

Our goal is to describe the boundary maps of the CW-complex generated by  $f$  in terms of the trajectories of the flow  $\varphi$ . If we assume a Smale-like condition (Definition 5) for the function  $f$  then the number of trajectories connecting a pair of critical points whose indices differ by one is finite. Furthermore, if  $M$  is orientable and the index of  $p_2$  is one greater than that of  $p_1$ , then we can associate to each trajectory  $\tau$  connecting  $p_2$  to  $p_1$  a sign  $+1$  or  $-1$ . This sign is determined by

comparing, via the flow  $\varphi$ , orientations on  $U(p_1)$  and  $U(p_2)$ . The degree of the pair  $(p_2, p_1)$  is

$$\text{degree}(p_2, p_1) = \sum_{\tau} \text{sign}(\tau),$$

where the sum is over all trajectories connecting  $p_2$  to  $p_1$ .

If  $P_1$  and  $P_2$  are the cells in the CW-complex corresponding to the critical points  $p_1$  and  $p_2$ , then the degree of the attaching map of  $P_2$  along  $P_1$  is the sum of the signs of the trajectories from  $p_2$  to  $p_1$ .

Now suppose  $M$  is a compact oriented manifold with corners and  $f : M \rightarrow \mathbb{R}$  is a Morse function satisfying the conditions of Definition 5. The goal of this article is to define a chain complex which is determined by the function  $f$  and whose homology is that of  $M$ . Let  $C_\ell(f)$  be the set of essential critical points of  $f$  with index  $j$ , and  $V_\ell$  the free Abelian group of formal  $\mathbb{Z}$ -linear combinations of the elements of  $C_\ell(f)$ . We can define  $\partial_\ell : V_\ell \rightarrow V_{j-1}$  by setting

$$\partial_\ell(p) = \sum_{q \in C_{j-1}(f)} \text{degree}(p, q)q$$

for each  $p \in C_\ell(f)$ , and extending linearly to all of  $V_\ell$ . Finally, we can state the

**Main Theorem. (Theorem 11)** *The free Abelian groups  $V_\ell$  and maps  $\partial_\ell$  form a chain complex whose homology groups are identical to the  $\mathbb{Z}$ -homology groups of the topological space  $M$ .*

### 3. THE ATTACHING MAP OF A CELL IN THE MORSE COMPLEX

Let  $p_1$  and  $p_2$  be essential critical points with indices  $\lambda_1$  and  $\lambda_2$  respectively. Let  $c_i = f(p_i)$  for  $i = 1, 2$ . We can choose  $\varepsilon$  sufficiently small that the set

$$U_\varepsilon(p_i) = U(p_i) - f^{-1}((c_i - \varepsilon]) = U(p_i) - M_{c_i - \varepsilon}$$

is homeomorphic to the interior of  $e^{\lambda_i}$ , a cell with dimension  $\lambda_i$ , for  $i = 1, 2$ . Here, we follow the convention that  $M_a = f^{-1}((-\infty, a])$ . In addition, suppose that  $f(p_1) = c_1$  and  $f(p_2) = c_2$ , where  $c_1 < c_2$ , and that there are no other critical points in  $f^{-1}([c_1, c_2])$ .

The flow  $\varphi$  can be viewed as a deformation retraction, and used to define homotopies

$$M_{c_2 - \varepsilon} \simeq \varphi(t, M_{c_2 - \varepsilon})$$

and

$$(3.1) \quad M_{c_2 - \varepsilon} \cup U(p_2) \simeq \varphi(t, M_{c_2 - \varepsilon} \cup U(p_2))$$

Since  $U(p_2)$  is invariant under the flow  $\varphi$ , it follows that

$$(3.2) \quad \varphi(t, M_{c_2 - \varepsilon} \cup U(p_2)) = \varphi(t, M_{c_2 - \varepsilon}) \cup U(p_2).$$

The space on the left hand side of (3.1) is  $M_{c_2 - \varepsilon}$  with a  $\lambda_2$ -cell attached, but on the right hand side of (3.2),  $U(p_2) - \varphi(t, M_{c_2 - \varepsilon})$  may not be homeomorphic to an open cell. This is a notable difference from the classical theory.

To simplify our thinking, we can consider  $M_{c_2 - \varepsilon} \cup U(p_2)$  to be  $M_{c_2 - \varepsilon}$  with a  $\lambda_2$ -cell attached via the inclusion map

$$\sigma : \partial U_\varepsilon(p_2) \hookrightarrow M_{c_2 - \varepsilon}.$$

Then the family of homotopy equivalences

$$M_{c_2-\varepsilon} \simeq \varphi(t, M_{c_2-\varepsilon}),$$

gives rise to homotopy equivalences

$$(3.3) \quad M_{c_2-\varepsilon} \cup_{\sigma} e^{\lambda_2} \simeq \varphi(t, M_{c_2-\varepsilon}) \cup_{\varphi(t, \cdot) \circ \sigma} e^{\lambda_2}.$$

Here we have identified  $\overline{U_{\varepsilon}(p_2)}$  with  $e^{\lambda_2}$ .

In the first case (3.2), we allow the whole space to flow by  $\varphi$ . The equality follows from the  $\varphi$ -invariance of  $U(p_2)$ . In the second case (3.3), we simply allow  $M_{c_2-\varepsilon}$  to flow by  $\varphi$ , and with it the image of the attaching map  $\sigma_2$ . (c.f. Lemma 3.7 in [Mi].)

From here it is a short step to prove

**Lemma 2.** *Let  $p_1$  and  $p_2$  be two essential critical points of  $f$  such that  $f(p_1) = c_1$ ,  $f(p_2) = c_2$ ,  $c_1 < c_2$  and there are no other critical points in  $f^{-1}([c_1 - \varepsilon, c_2])$ . Then there is a map  $\Phi_2 : M_{c_2-\varepsilon} \rightarrow M_{c_1+\varepsilon}$  such that*

$$M_{c_2-\varepsilon} \cup U(p_2) = M_{c_2-\varepsilon} \cup_{\sigma} e^{\lambda_2} \simeq M_{c_1+\varepsilon} \cup_{\Phi_2 \circ \sigma} e^{\lambda_2},$$

where  $\lambda_2$  is the index of  $p_2$ . Moreover,  $\Phi_2$  may be defined in a way that preserves  $\varphi$ -trajectories in the sense that for every  $q \in M_{c_2-\varepsilon}$ , the points  $q$  and  $\Phi_2(q)$  lie on the same  $\varphi$ -trajectory.

*Proof.* The equality follows immediately from the discussion above. To define the map  $\Phi_2$ , we use the flow  $\varphi$  to construct a homotopy. For each  $q \in M_{c_2-\varepsilon}$ , define

$$t_q^{(1)} = \inf \{t \in \mathbb{R}^+ : \varphi(t, q) \in M_{c_1+\varepsilon}\}$$

The function  $q \mapsto t_q^{(1)}$  is continuous. Since there are no critical values between  $c_1$  and  $c_2$ ,  $t_q^{(1)}$  is finite for each  $q \in M_{c_2-\varepsilon}$ . Moreover, since  $M_{c_2-\varepsilon}$  is compact,  $T_1 = \sup\{t_q^{(1)} : q \in M_{c_2-\varepsilon}\}$  is finite.

Now we can define a homotopy  $H_1 : [0, 1] \times M_{c_2-\varepsilon} \rightarrow M_{c_1+\varepsilon}$  by

$$H_1(s, q) = \begin{cases} \varphi(T_1 s, q) & T_1 s \leq t_q^{(1)} \\ \varphi(t_q^{(1)}, q) & T_1 s \geq t_q^{(1)} \end{cases}$$

If we set  $\Phi_2(q) = H_1(1, q)$  for  $q \in M_{c_2-\varepsilon}$ , the homotopy equivalence in the lemma follows. q.e.d.

$\Phi_2$  can be interpreted as allowing each point to flow by  $\varphi$  until it reaches  $M_{c_1+\varepsilon}$ . Now we must deal with the more complicated situation in the neighborhood of the critical point  $p_1$ .

**Lemma 3.** *Let  $p_1$  be an essential critical point of  $f$  such that  $f(p_1) = c_1$  and let  $\varepsilon$  be small enough that there are no other critical points in  $f^{-1}([c_1 - \varepsilon, c_1 + \varepsilon])$ . Then there is a homotopy equivalence*

$$\Psi_2 : M_{c_1+\varepsilon} \rightarrow M_{c_1-\varepsilon} \cup U(p_1).$$

Moreover,  $\Psi_2$  can be chosen such that  $\Psi_2(q) = p_1$  if and only if  $q \in S(p_1) \cap M_{c_1+\varepsilon}$ .

*Proof.* Since  $p_1$  is an essential critical point with index  $\lambda_1$ , we can choose a coordinate chart such that

$$f(\mathbf{x}) = c_1 - x_1^2 - \cdots - x_{\lambda_1}^2 + x_{\lambda_1+1}^2 + \cdots + x_{n-j}^2 + x_{n-j+1} + \cdots + x_n$$

We can define, for  $\varepsilon$  sufficiently small, a region  $H$  and show that  $M_{c_1-\varepsilon} \cup H$  is homotopy equivalent to both  $M_{c_1+\varepsilon}$  and  $M_{c_1-\varepsilon} \cup U(p_1)$ . To do so we first choose a smooth function  $\mu : \mathbb{R} \rightarrow \mathbb{R}$  so that  $\mu(0) > \varepsilon$ ,  $\mu(r) = 0$  for  $r > 2\varepsilon$ , and  $-1 < \mu' \leq 0$ . Then we set

$$F(\mathbf{x}) = f(\mathbf{x}) - \mu(x_1^2 + \cdots + x_{\lambda_1}^2 + 2(x_{\lambda_1+1}^2 + \cdots + x_{n-j}^2) + 2(x_{n-j+1} + \cdots + x_n))$$

Finally, we define  $H = \overline{F^{-1}(-\infty, c_1 - \varepsilon]} - M_{c_1-\varepsilon}$ . Note that  $H$  is used to denote this space, as it is in [Mi], despite the possibility of confusion with the homotopies defined above and below. In this article,  $H_i$  is a homotopy and  $H$  is a subset of  $M$ .

The proof of the lemma will be in two parts. First, we construct a homotopy to show that  $M_{c_1+\varepsilon} \simeq M_{c_1-\varepsilon} \cup H$ . We then construct a different homotopy to show that  $M_{c_1-\varepsilon} \cup H \simeq M_{c_1-\varepsilon} \cup U(p_1)$ .

For each  $q \in M_{c_1+\varepsilon}$ , define

$$t_q^{(2)} = \inf \{t \in \mathbb{R}^+ : \varphi(t, q) \in M_{c_1-\varepsilon} \cup H\}$$

The function  $q \mapsto t_q^{(2)}$  is continuous, and  $t_q^{(2)}$  is finite for each  $q \in M_{c_2-\varepsilon}$ . Moreover, since  $M_{c_2-\varepsilon}$  is compact,  $T_2 = \sup\{t_q^{(2)} : q \in M_{c_2-\varepsilon}\}$  is finite.

Now we can define the first of our two homotopies  $H_2 : [0, 1] \times M_{c_1+\varepsilon} \rightarrow M_{c_1-\varepsilon} \cup H$  by

$$H_2(s, q) = \begin{cases} \varphi(T_2 s, q) & T_2 s \leq t_q^{(2)} \\ \varphi(t_q^{(2)}, q) & T_2 s \geq t_q^{(2)} \end{cases}$$

This shows that  $M_{c_1+\varepsilon} = H_2(0, M_{c_1+\varepsilon}) \simeq H_2(1, M_{c_1+\varepsilon}) = M_{c_1-\varepsilon} \cup H$ .

Now we show that  $M_{c_1-\varepsilon} \cup H \simeq M_{c_1-\varepsilon} \cup U(p_1)$ . In our chosen coordinate system the stable set of  $p_1$  corresponds to

$$\{(x_1, \dots, x_n) | x_1 = \cdots = x_\lambda = 0\}.$$

The unstable set of  $p_1$  corresponds to

$$\{(x_1, \dots, x_n) | x_{\lambda+1} = \cdots = x_n = 0\}.$$

Consider the function  $g(\mathbf{x}) = f(p_1) + x_{\lambda+1}^2 + \cdots + x_{n-j}^2 + x_{n-j+1} + \cdots + x_n$ . This function is constant on the unstable set, with a value of  $f(p_1)$ . For  $q$  outside the unstable set,  $f(q) > f(p_1)$ . Moreover, on the stable set the  $-\nabla g$  agrees with  $-\nabla f$ .

Let  $X$  denote the modified (negative) gradient vector field of the function  $g$ . Then  $X(q) = G(q)$  for  $q \in S(p_1)$ , and  $X(q) = 0$  for  $q \in U(p_1)$ . The flow of  $X$  is directed toward  $U(p_1)$ . Let  $\psi$  denote the flow of the vector field  $\frac{X}{\|X\|}$ , and note that  $\psi$  preserves the  $\varphi$ -trajectories that have  $p_1$  as a lower endpoint.

For each  $q$  in  $H$ , we can set  $t_q^{(3)} = \inf\{t \in \mathbb{R}^+ : \psi(t, q) \in M_{c_1-\varepsilon} \cup U(p_1)\}$ , and for  $q \in M_{c_1-\varepsilon}$  set  $t_q^{(3)} = 0$ . Then define  $T_3 = \sup\{t_q^{(3)} : q \in H\}$ . Then the homotopy  $H_3 : [0, 1] \times M_{c_1-\varepsilon} \cup H \rightarrow M_{c_1-\varepsilon} \cup U(p_1)$  given by

$$H_3(s, q) = \begin{cases} \psi(T_3 s, q) & T_3 s \leq t_q^{(3)} \\ \psi(t_q^{(3)}, q) & T_3 s \geq t_q^{(3)} \end{cases}$$

gives a homotopy equivalence  $H_3(1, \cdot) : M_{c_1-\varepsilon} \cup H \rightarrow M_{c_1-\varepsilon} \cup U(p_1)$ .

Then the composition  $\Psi_2 = H_3(1, \cdot) \circ H_2(1, \cdot)$  is a homotopy equivalence  $\Psi_2 : M_{c_1+\varepsilon} \rightarrow M_{c_1-\varepsilon} \cup U(p_1)$ .

q.e.d.

There is no reason, in principle, that the  $\varepsilon$  used to define  $U_\varepsilon(p_2)$  must be the same as the  $\varepsilon$  in Lemma 2 and Lemma 3. Nor must the  $\varepsilon$ 's from the two Lemmas agree with each other. They can be chosen to be the same, however, and this serves to simplify the notation in the remainder of our argument.

Now let's view this in terms of attaching maps. Let  $\sigma_2$  denote the inclusion map  $\sigma_2 : \partial U_\varepsilon(p_2) \rightarrow M_{c_2-\varepsilon}$ . (Previously this was simply  $\sigma$ .) Then we can use the homotopy equivalence  $\Phi_2$  to define an attaching map

$$\Phi_2 \circ \sigma_2 : \partial U_\varepsilon(p_2) \rightarrow M_{c_1+\varepsilon}.$$

Now, identifying  $U_\varepsilon(p_2)$  with  $e^{\lambda_2}$ , we find that

$$M_{c_2+\varepsilon} \simeq M_{c_2-\varepsilon} \cup U(p_2) \simeq M_{c_1+\varepsilon} \cup_{\Phi_2 \circ \sigma_2} e^{\lambda_2}.$$

The first homotopy equivalence follows from Lemma 3.

Using the homotopy equivalence  $\Psi_2$ , we can define an attaching map

$$\Psi_2 \circ \Phi_2 \circ \sigma_2 : \partial U_\varepsilon(p_2) \rightarrow M_{c_1-\varepsilon} \cup U(p_1).$$

From this, it follows that

$$M_{c_2+\varepsilon} \simeq M_{c_2-\varepsilon} \cup U(p_2) \simeq M_{c_1-\varepsilon} \cup U(p_1) \cup_{\Psi_2 \circ \Phi_2 \circ \sigma_2} e^{\lambda_2}.$$

Let  $\Sigma_2 = \Psi_2 \circ \Phi_2 \circ \sigma_2$ . Because  $\Phi_2$  preserves all the  $\varphi$ -trajectories and  $\Psi_2$  preserves those in  $S(p_1)$ , it follows that  $\Sigma_2(q) = p_1$  if and only if  $q \in \partial U_\varepsilon(p_2)$  lies on a trajectory connecting  $p_2$  to  $p_1$ .

Using these ideas, we can construct a cellular complex (which will later prove to be a CW-complex) that is homotopy equivalent to  $M$ . Let  $p_1, p_2, \dots, p_m$  be the essential critical points of  $M$ , each  $p_i$  having index  $\lambda_i$ . Assume the values  $f(p_i)$  are distinct and the essential critical points are labeled such that  $f(p_i) < f(p_j)$  whenever  $i < j$ . Then  $c_1 = f(p_1)$  must be an absolute minimum, and  $M_{c_1+\varepsilon}$  is homotopy equivalent to the 0-cell,  $U_\varepsilon(p_1)$ . In fact the map  $\Psi_2$  is a homotopy equivalence.

Now, as in Lemma 2 and Lemma 3, we consider the inclusion map  $\sigma_2 : \partial U_\varepsilon(p_2) \hookrightarrow M_{c_2-\varepsilon}$ . Viewing  $U_\varepsilon(p_2)$  as a  $\lambda_2$ -cell, and using the attaching map  $\Sigma_2 = \Psi_2 \circ \Phi_2 \circ \sigma_2$ , of  $U_\varepsilon(p_2)$  onto  $U_\varepsilon(p_1) = \{p_1\}$  we see that

$$\begin{aligned} M_{c_2+\varepsilon} &\simeq U_\varepsilon(p_1) \cup_{\Sigma_2} U_\varepsilon(p_2) \\ &\simeq e^{\lambda_1} \cup_{\Sigma_2} e^{\lambda_2}. \end{aligned}$$

If  $\varepsilon$  is sufficiently small, then  $U_\varepsilon(p_3)$  is homeomorphic to a  $\lambda_3$ -cell. It takes only a slight extension to define the attaching map for this cell. Let  $\sigma_3$  be the inclusion map  $\partial U_\varepsilon(p_3) \hookrightarrow M_{c_3-\varepsilon}$ , and define  $\Phi_3, \Psi_3$  for the critical points  $p_2$  and  $p_3$  using Lemma 2 and Lemma 3. Then  $\tilde{\Sigma}_3 = \Psi_3 \circ \Phi_3 \circ \sigma_3$  gives the appropriate attaching map of  $\partial U_\varepsilon(p_3)$  onto  $M_{c_2-\varepsilon} \cup U(p_2) \simeq M_{c_2-\varepsilon} \cup_{\sigma_2} e^{\lambda_2}$ .

Since  $\Psi_2 \circ \Phi_2$  is a homotopy equivalence from  $M_{c_2-\varepsilon} \cup_{\sigma_2} e^{\lambda_2}$  to  $e^{\lambda_1} \cup_{\Sigma_2} e^{\lambda_2}$ , the desired attaching map is given by

$$\Sigma_3 = \Psi_2 \circ \Phi_2 \circ \Psi_3 \circ \Phi_3 \circ \sigma_3.$$

It follows that

$$\begin{aligned} M_{c_3+\varepsilon} &\simeq U_\varepsilon(p_1) \cup_{\Sigma_2} U_\varepsilon(p_2) \cup_{\Sigma_3} U_\varepsilon(p_3) \\ &\simeq e^{\lambda_1} \cup_{\Sigma_2} e^{\lambda_2} \cup_{\Sigma_3} e^{\lambda_3}. \end{aligned}$$

Because of the construction of  $\Phi_3$  and  $\Psi_3$ ,  $\Sigma_3(q) = p_2$  if and only if  $q \in \partial U_\varepsilon(p_3)$  lies on a  $\varphi$ -trajectory connecting  $p_3$  to  $p_2$ . We would like to be able to say that



$\Sigma_3(q) = p_1$  if and only if  $q \in \partial U_\varepsilon(p_3)$  lies on a  $\varphi$ -trajectory connecting  $p_3$  to  $p_1$ . This will be true provided that  $\Psi_3$  does not disrupt these trajectories.

This can not be true in every case. With care, though, we can be sure that trajectories connecting critical points whose indices differ by one are not disturbed. This issue is resolved by Lemmas 9 and 10.

In general, for each critical point  $p_k$  we define

$$\Sigma_k = \Psi_2 \circ \Phi_2 \circ \dots \circ \Psi_k \circ \Phi_k \circ \sigma_k.$$

These maps gives a homotopy equivalence

$$\begin{aligned} M &\simeq U_\varepsilon(p_1) \cup_{\Sigma_2} U_\varepsilon(p_2) \cup \dots \cup_{\Sigma_m} U_\varepsilon(p_m) \\ &\simeq e^{\lambda_1} \cup_{\Sigma_2} e^{\lambda_2} \cup \dots \cup_{\Sigma_m} e^{\lambda_m}. \end{aligned}$$

We shall see that under appropriate conditions (Definition 5), this cellular complex is in fact a CW-complex.

#### 4. THE DEGREE OF AN ATTACHING MAP

Our goal is to show that, as in classical Morse theory, the degree of this attaching map can be determined by counting the trajectories connecting essential critical points. One difficulty is that, although these attaching maps are continuous, they are in general not smooth. Lemmas 4, 5 and 6 show that the maps possess enough regularity that we can establish a Smale-like transversality condition (Definition 5). Using this definition, Lemmas 7 and 8 show that a finite number of trajectories connect critical points whose indices differ by one. Finally a after a careful consideration of orientations along each trajectory connecting critical points, we define a degree to each such pair.

**Lemma 4.** *Let  $K$  be a stratum with dimension  $k$  and  $L$  be a stratum in the closure of  $K$  with dimension  $k - 1$ . Let  $p \in K$  and suppose that  $\varphi(s, p) \in K$  for  $s < t_p$  and  $\varphi(s, p) \in L$  for  $t_p < s \leq T$ . Then there is a neighborhood  $U \subset K$  of  $p$  such that  $\varphi(T, \cdot) : U \rightarrow M$  is smooth.*

*Proof.* Choose a coordinate system near  $\varphi(t_p, p)$  such that for  $q \in K$ ,  $x_{k+1}(q) = \dots = x_n(q) = 0$  and for  $q \in L$ ,  $x_k(q) = \dots = x_n(q) = 0$ . We know that the  $k$ th coordinate satisfies  $\varphi_k(t_p, p) = 0$ , and in addition

$$\left. \frac{\partial \varphi_k}{\partial t} \right|_{(t_p, p)} \neq 0,$$

by the definition of Morse function.

From the Implicit Function Theorem, it follows that there is an open set  $U \subset K$  containing  $p$  and an open set  $V \subset \mathbb{R}$  such that for each  $q \in U$  there is a unique  $g(q) \in V$  such that  $\varphi_k(g(q), q) = 0$ . Moreover, the function  $g$  is smooth.

We can use this function  $g$  to define two additional functions:  $\gamma : U \rightarrow \mathbb{R} \times U$  given by  $\gamma(q) = (g(q), q)$  and  $\tau : \gamma(U) \rightarrow \mathbb{R} \times M$  given by  $\tau(t, q) = (T - t, \varphi(t, q))$ . Since  $\varphi$  is smooth whenever it remains in a single stratum, these are themselves smooth functions. We can write

$$\varphi(T, q) = (\varphi \circ \tau \circ \gamma)(q),$$

a composition of smooth functions, so  $\varphi(T, \cdot) : U \rightarrow M$  is smooth.

q.e.d.

**Lemma 5.** *Let  $K$  be a stratum with dimension  $k - 1$  in the closure of a stratum  $L$  with dimension  $k$ . Let  $p \in K$  and suppose that  $\varphi(s, p) \in K$  for  $s \leq t_p$  and  $\varphi(s, p) \in L$  for  $t_p < s \leq T$ . Then there is a neighborhood  $U \subset K$  of  $p$  such that  $\varphi(T, \cdot) : U \rightarrow M$  is smooth.*

*Proof.* Choose a coordinate system near  $\varphi(t_p, p)$  such that for  $q \in K$ ,  $x_k(q) = \dots = x_n(q) = 0$  and for  $q \in L$ ,  $x_{k+1}(q) = \dots = x_n(q) = 0$ . Let  $h(s, q) = (-\nabla f)_k(\varphi(s, q))$ . Then  $h$  is a smooth real valued function. Since the trajectory  $\varphi(\cdot, p)$  moves from  $K$  to  $L$  at time  $t_p$ , we must have  $h(t_p, p) = 0$ .

By the definition of a Morse function, we know that

$$\left. \frac{\partial h}{\partial t} \right|_{(t_p, p)} \neq 0.$$

It then follows from the Implicit Function Theorem, that there is an open set  $U \subset K$  containing  $p$  and an open subset  $V \subset \mathbb{R}$  such that for each  $q \in U$  there is a unique  $g(q) \in V$  such that  $h(g(q), q) = 0$ . Moreover, the function  $g$  is smooth.

As before, we can define smooth functions  $\gamma : U \rightarrow \mathbb{R} \times U$  given by  $\gamma(q) = (g(q), q)$  and  $\tau : \gamma(U) \rightarrow \mathbb{R} \times M$  given by  $\tau(t, q) = (T - t, \varphi(t, q))$  and write

$$\varphi(T, q) = (\varphi \circ \tau \circ \gamma)(q),$$

Again, this is a composition of smooth functions, and so  $\varphi(T, \cdot) : U \rightarrow M$  is smooth. q.e.d.

Now let  $p_2$  be an essential critical point with index  $\lambda_2 = j$  and  $p_1$  an essential critical point with index  $\lambda_1 = j - 1$ . Suppose that  $S(p_1) \cap U(p_2) \neq \emptyset$ . Choose a

$$q \in S(p_1) \cap U(p_2) \cap M_{c_1 + \varepsilon}.$$

Then there is a  $q_1 \in U_\varepsilon(p_2)$  and a time  $T$  such that  $q = \varphi(T, q_1)$ .

**Lemma 6.** *If the trajectory  $\varphi(s, q_1)$  only ever changes between strata whose difference in dimension is 1, then there is a neighborhood  $V$  in the stratum containing  $q_1$  such that  $\varphi(T, \cdot) : V \rightarrow M$  is smooth.*

*Proof.* The result follows from repeated application of Lemma 4 and Lemma 5. Since  $f^{-1}(-\infty, r]$  is compact for each choice of  $r$ , there can be only finitely many stratum changes ([Han]). For each stratum change,  $i = 1, \dots, N$ , we get a corresponding neighborhood  $V_i$ . Since there are finitely many, we may choose

$$V = V_1 \cap \dots \cap V_N.$$

q.e.d.

**Definition 5.** We say that the Morse function  $f$  is a *Morse-Smale function* if for each pair of critical points,  $p$  and  $p'$  with  $f(p) > f(p')$ , and for  $\varepsilon$  sufficiently small the parametrized space  $U(p)$  and the set  $S_\varepsilon(p') = S(p') \cap M_{f(p') + \varepsilon}$  intersect transversely in the following sense: if  $q \in S_\varepsilon(p') \cap U(p)$ ,  $K$  is the stratum containing  $q$ , and  $q$  has a neighborhood  $N \subset M$  such that  $S_\varepsilon(p') \cap U(p) \cap N = S_\varepsilon(p') \cap U(p) \cap K$ , then  $U(p)$  and  $S_\varepsilon(p') \cap K$  intersect transversely in  $K$ .

Let  $p_i$  and  $p_j$  be two critical points with indices  $\lambda_i$  and  $\lambda_j$ , respectively, satisfying  $\lambda_j - \lambda_i = 1$ . Let  $\mathfrak{T}$  be the set of trajectories from  $p_j$  to  $p_i$ . We wish to show that the transversality condition in Definition 5 together with the compactness of  $M$  ensure that  $\mathfrak{T}$  is a finite set.

We begin with the following

**Lemma 7.** *Suppose that  $\widetilde{M}$  is an  $m$ -manifold,  $\widetilde{S}$  a  $k$ -dimensional regular submanifold. Let  $\widetilde{U}$  be an  $(m-k)$ -manifold, and  $\Phi : \widetilde{U} \rightarrow \widetilde{M}$  a smooth map such that  $\widetilde{S} \pitchfork \Phi(\widetilde{U})$ . Then  $\Phi^{-1}(\widetilde{S} \cap \Phi(\widetilde{U}))$  consists of isolated points in  $\widetilde{U}$ .*

*Proof.* First, since  $\dim(\widetilde{S}) + \dim(\Phi(\widetilde{U})) = k + (m-k) = m$ , the transverse intersection of  $\widetilde{S}$  and  $\Phi(\widetilde{U})$  is zero dimensional. Let  $q \in \widetilde{U}$  such that  $\Phi(q) \in \widetilde{S}$ . We will show that there is a neighborhood  $N_q$  of  $q$  such that  $\Phi^{-1}(\widetilde{S}) \cap N_q = \{q\}$ .

let  $\mathbf{x} = (x^1, \dots, x^m)$  be a coordinate chart on  $\widetilde{M}$  such that  $\mathbf{x}(\Phi(q)) = (0, \dots, 0)$ , and  $(x^1, \dots, x^k)$  gives a coordinate chart on  $\widetilde{S}$ . Then for  $p \in \widetilde{S}$

$$x^i(p) = 0, \quad \text{for } i = k+1, \dots, m.$$

Let  $\bar{\mathbf{x}} = (x^{k+1}, \dots, x^m)$ . We want to show that  $\bar{\mathbf{x}} = \bar{\mathbf{x}} \circ \Phi$  is a coordinate chart on  $\widetilde{U}$ . It suffices to show that if  $\mathbf{y}$  is a coordinate chart on  $\widetilde{U}$ , then  $D(\bar{\mathbf{x}} \circ \mathbf{y}^{-1})$  is non-singular.

Since  $\widetilde{S} \pitchfork \Phi(\widetilde{U})$ , the vectors

$$\left\{ \frac{\partial}{\partial x^1} \Big|_{\Phi(q)}, \dots, \frac{\partial}{\partial x^k} \Big|_{\Phi(q)}, \Phi_* \left( \frac{\partial}{\partial y^{k+1}} \Big|_q \right), \dots, \Phi_* \left( \frac{\partial}{\partial y^m} \Big|_q \right) \right\}$$

form a basis for  $T_{\Phi(q)}(\widetilde{M})$ , as do  $\left\{ \frac{\partial}{\partial x^1} \Big|_{\Phi(q)}, \dots, \frac{\partial}{\partial x^m} \Big|_{\Phi(q)} \right\}$ . We can thus write

$$\Phi_* \left( \frac{\partial}{\partial y^i} \right) = \sum_{j=1}^n \alpha_i^j \frac{\partial}{\partial x^j} \quad \text{for } i = k+1, \dots, n.$$

Using this expression, it is clear that the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ \alpha_{k+1}^1 & & \cdots & \alpha_{k+1}^m \\ \vdots & & & \vdots \\ \alpha_m^1 & & \cdots & \alpha_m^m \end{bmatrix} = \left[ \begin{array}{c|ccc} I_k & & 0 & \\ \hline & \alpha_{k+1}^{k+1} & \cdots & \alpha_{k+1}^m \\ * & \vdots & \ddots & \vdots \\ & \alpha_m^{k+1} & \cdots & \alpha_m^m \end{array} \right] = \left[ \begin{array}{c|c} I_k & 0 \\ \hline * & M \end{array} \right]$$

is non-singular (having linearly independent rows). It follows that the matrix  $M$  is non-singular. But  $M$  is the transpose of the matrix for

$$D(\bar{\mathbf{x}} \circ (\Phi \circ \mathbf{y}^{-1})) = D((\bar{\mathbf{x}} \circ \Phi) \circ \mathbf{y}^{-1}) = D(\bar{\mathbf{x}} \circ \mathbf{y}^{-1}),$$

evaluated at  $\mathbf{y}(q)$ , so  $D(\bar{\mathbf{x}} \circ \mathbf{y}^{-1})$  is non-singular in a neighborhood of  $\mathbf{y}(q)$ .

Now, since  $\bar{\mathbf{x}}$  is a coordinate chart in a neighborhood of  $q$ , there is an open set  $N_q$  of  $q$  such that for  $q' \in N_q$  with  $q' \neq q$ ,  $\bar{\mathbf{x}}(q') \neq (0, \dots, 0)$ . So  $\Phi(q') \notin \widetilde{S}$ .

q.e.d.

**Lemma 8.** *Let  $f : M \rightarrow \mathbb{R}$  be a Morse-Smale function on a manifold with corners  $M$ . Let  $p_i$  and  $p_j$  be two critical points of  $f$  with indices  $\lambda_i$  and  $\lambda_j$  respectively, satisfying  $\lambda_j - \lambda_i = 1$ . Then the set  $\mathfrak{T}$  of trajectories from  $p_j$  to  $p_i$  is finite.*

*Proof.* Let  $c_i = f(p_i)$  and  $c_j = f(p_j)$ . Then  $\partial U_\varepsilon(p_j)$  is diffeomorphic to  $S^{\lambda_j-1}$ . For each element of  $\tau \in \mathfrak{T}$  there is exactly one  $q_\tau \in \partial U_\varepsilon(p_j) \cap \tau$ .

Suppose that  $\mathfrak{T}$  is an infinite set. Then, since  $\partial U_\varepsilon(p_j)$  is compact, there is a convergent sequence  $\{q_i\}_{i=1}^\infty$  such that  $q_i \in \partial U_\varepsilon(p_j) \cap \tau_i$  for some  $\tau_i \in \mathfrak{T}$ . Let  $q_0 = \lim_{i \rightarrow \infty} q_i$ .

We wish to show that  $q_0 = \partial U_\varepsilon(p_j) \cap \tau_0$  for some  $\tau_0 \in \mathfrak{T}$ . Choose  $T$  such that for  $p = \varphi(T, q_0)$ ,  $f(p) < c_i + \varepsilon$ . Then for  $i$  sufficiently large  $f(\varphi(T, q_i)) < c_i + \varepsilon$ , and since for each  $i$   $\lim_{t \rightarrow \infty} \varphi(t, q_i) = p_i$ ,  $\varphi(T, q_i) \in S_\varepsilon(p_i)$  for  $i$  sufficiently large. Now,  $\varphi$  is continuous, and  $S_\varepsilon(p_i)$  is closed, so  $p = \lim_{i \rightarrow \infty} \varphi(T, q_i) \in S_\varepsilon(p_i)$ . It follows that  $q_0$  lies on some trajectory connecting  $p_j$  to  $p_i$ .

Now we will show that  $p_0$  is isolated in  $\partial U_\varepsilon \cap \mathfrak{T}$ , contradicting the choice of  $p_0$ . Let  $K$  be the stratum containing  $p = \varphi(T, q_0)$  and set  $c = f(p)$ . Let  $\widetilde{M} = f^{-1}(c) \cap K$ ,  $\widetilde{S} = S_\varepsilon(p_i) \cap \widetilde{M}$ . Assume there is a neighborhood  $U_0$  of  $q_0$  in  $M$  on which  $\varphi(T, \cdot)$  is smooth. (If there is not, replace  $T$  with a slightly larger  $T'$ .) For each  $q \in U_0$  there is a trajectory of  $\varphi$  containing  $q$ . We will denote this trajectory by  $\gamma_q$ . For a given  $q \in U_0$ , let  $\tilde{q} = \gamma_q \cap \partial U_\varepsilon(p_j)$ ,  $\bar{q} = \gamma_q \cap \varphi(T, \cdot)^{-1}(\widetilde{S})$ , and let  $t_q \in \mathbb{R}$  such that

$$\begin{aligned} \varphi(t_q, \tilde{q}) &= \bar{q} & \text{if } f(\tilde{q}) \geq f(\bar{q}) \\ \varphi(-t_q, \bar{q}) &= \tilde{q} & \text{if } f(\tilde{q}) \leq f(\bar{q}). \end{aligned}$$

The maps  $q \mapsto \tilde{q}$ ,  $q \mapsto \bar{q}$  and  $q \mapsto t_q$  are all smooth, and so the function  $\Phi_0 : U_0 \rightarrow M$  given by  $\Phi_0(q) = \varphi(T + t_q, q)$  is smooth. Let  $\widetilde{U} = \partial U_\varepsilon(p_j) \cap U_0$ , and  $\Phi = \Phi_0|_{\widetilde{U}} : \widetilde{U} \rightarrow \widetilde{M}$ .

We can let  $k = \dim(\widetilde{S}) = \dim(K) - \lambda_i - 1$  and  $m = \dim(\widetilde{M}) = \dim(K) - 1$ . Then  $\dim(\partial U_\varepsilon(p_j)) = \lambda_j - 1 = m - k$ . Choose a coordinate chart at  $p \in \widetilde{M}$ ,  $\mathbf{x} = (x^1, \dots, x^m)$ , such that  $(x^1, \dots, x^k)$  is a coordinate chart on  $\widetilde{S}$  and

$$x^i(p') \neq 0 \quad \text{whenever } p' \notin \widetilde{S} \text{ and } i > k.$$

This can be done in such a way that  $\frac{\partial}{\partial x^k}|_p$  is tangent to the trajectory  $\tau_0$ .

Now choose a coordinate chart  $\mathbf{y} = (y^{k+1}, \dots, y^{m+1})$  on  $U_0$ , such that  $\mathbf{y} = (y^{k+1}, \dots, y^m)$  is a coordinate chart on  $\widetilde{U}$  and  $\frac{\partial}{\partial y^{m+1}}|_{q_0}$  is tangent to the trajectory  $\tau_0$ . Then

$$\Phi_* \left( \frac{\partial}{\partial y^i} |_{q_0} \right) = \Phi_{0*} \left( \frac{\partial}{\partial y^i} |_{q_0} \right) \in T_p \widetilde{M} \text{ for } i = k, \dots, m.$$

In addition,  $\Phi_{0*}(\frac{\partial}{\partial y^n}|_{q_0})$  will be tangent (at  $p$ ) to the trajectory  $\tau_0$ .

Since  $f$  is a Morse-Smale function, the vectors

$$\left\{ \frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^{k+1}}|_p, \Phi_{0*} \left( \frac{\partial}{\partial y^{k+1}} |_{q_0} \right), \dots, \Phi_{0*} \left( \frac{\partial}{\partial y^{m+1}} |_{q_0} \right) \right\}$$

span  $T_p K$ . Since the vectors  $\frac{\partial}{\partial x^{k+1}}|_p$  and  $\Phi_{0*}(\frac{\partial}{\partial y^{m+1}}|_{q_0})$  are tangent to  $T_p \tau_0$ , and the remaining  $m = \dim(\widetilde{M})$  vectors are all in  $T_p \widetilde{M}$ ,

$$\left\{ \frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^k}|_p, \Phi_* \left( \frac{\partial}{\partial y^{k+1}} |_{q_0} \right), \dots, \Phi_* \left( \frac{\partial}{\partial y^m} |_{q_0} \right) \right\}$$

form a basis for  $T_p \widetilde{M}$ . The first  $k$  vectors of this basis are a basis of  $T_p \widetilde{S}$ , and the remaining  $m - k$  vectors are a basis for  $T_p \Phi(\partial U_\varepsilon(p_j))$ . It follows that

$$\widetilde{S} \cap \Phi(\partial U_\varepsilon(p_j)).$$

By Lemma 7,  $\Phi^{-1}(\widetilde{S} \cap \Phi(\widetilde{U}))$  consists of isolated points. Note that for sufficiently large  $i$ ,  $q_i \in \Phi^{-1}(\widetilde{S} \cap \Phi(\widetilde{U}))$  and in addition  $q_0 \in \Phi^{-1}(\widetilde{S} \cap \Phi(\widetilde{U}))$ . This, however, contradicts the fact that  $\lim_{i \rightarrow \infty} q_i = q_0$ . Consequently, the number of trajectories connecting  $p_j$  to  $p_i$  must be finite. q.e.d.

Now assume that  $M$  is an oriented manifold. At each essential critical point  $p$ , the tangent space  $T_p M$  can be decomposed into a stable space and an unstable space,  $T_p M = E^+(p) \oplus E^-(p)$ . Choose an orientation for the unstable space  $E^-(p)$ . This induces an orientation on  $E^+(p)$  in the following way:

Let  $\lambda_p$  be the index of  $p$ , and choose a basis  $v_1, \dots, v_{\lambda_p} \in E^-(p)$  which represents the orientation on  $E^-(p)$ . Then choose  $v_{\lambda_p+1}, \dots, v_n \in E^+(p)$  such that  $v_1, \dots, v_n$  is a basis which represents the orientation on  $T_p M$ . The vectors  $v_{\lambda_p+1}, \dots, v_n$  determine an orientation for  $E^+(p)$ .

Now for critical points  $p_i$  and  $p_j$ , the orientation on  $E^+(p_i)$  determines an orientation on  $S_\varepsilon(p_i)$ , which in turn determines an orientation on  $T_q S(p_i)$  for  $q \in S(p_i)$ . In a similar way, the choice of orientation on  $E^-(p_j)$  induces an orientation on  $T_{q'} U(p_j)$  for  $q' \in U_\varepsilon(p_j)$ .

Note that if the function  $f : M \rightarrow \mathbb{R}$  is a Morse-Smale function and  $\varphi(T, q') = q$ , then the derivative  $\varphi(T, \cdot)_*$  restricted to  $U(p_j)$  has full rank at  $\varphi(T, \cdot)^{-1}(q)$ .

Now the map  $\varphi(T, \cdot)_*$  pushes forward the orientation on  $T_{q'} U(p_j)$  to give an orientation on  $T_q \varphi(T, U(p_j))$ . Since  $\dim(\varphi(T, U(p_j))) = j$  and  $\dim(S_\varepsilon(p_i)) = n - j + 1$ , these two sets intersect in the one dimensional trajectory through  $q$ . We can choose a basis for  $T_q \varphi(T, U(p_j))$  which represents the induced orientation and such that the last vector is  $\frac{d}{dt}|_T \varphi(t, q_1)$ . We also may choose a basis representing the orientation on  $T_q S(p_i)$ , with  $\frac{d}{dt}|_T \varphi(t, q_1)$  as its first vector. When combined, these bases give an orientation for  $T_q M$ . If this orientation agrees with the chosen orientation on  $M$  then we say the sign of the trajectory from  $p_j$  to  $p_i$  through  $q_1$  is  $+1$ . Otherwise, the sign is  $-1$ .

Now if  $\mathfrak{T}$  is the set of trajectories from  $p_j$  to  $p_i$ , Lemma 8 ensures that  $\mathfrak{T}$  is a finite set. So, we can define

$$\text{degree}(p_j, p_i) = \sum_{\tau \in \mathfrak{T}} \text{sign}(\tau).$$

This degree, we shall see, is the degree of the attaching map  $\Sigma_j$  at  $p_i$ .

## 5. THE MORSE COMPLEX FOR A MORSE FUNCTION ON A MANIFOLD WITH CORNERS

**5.1. The Geometric Complex of a Morse Function.** Let  $P_k$  denote the cell  $U_\varepsilon(p_k)$ . We know that  $M$  is homotopy equivalent to the cellular complex

$$X = P_1 \cup_{\Sigma_2} P_2 \cup \dots \cup_{\Sigma_m} P_m.$$

Recall that  $\Sigma_j$  was defined by

$$\Sigma_j = \Psi_2(1, \cdot) \circ \Phi_2(1, \cdot) \circ \dots \circ \Psi_j(1, \cdot) \circ \Phi_j(1, \cdot) \circ \sigma_j,$$

which attaches  $P_j$  to  $P_1 \cup_{\Sigma_2} P_2 \cup \dots \cup_{\Sigma_{j-1}} P_{j-1}$ . The map  $\Sigma_j$  is defined by a sequence of maps derived from flows. They alternate between a  $\Phi_k$  which preserves the  $\varphi$ -trajectories, and a  $\Psi_k$  which does not.

We wish to show that this cellular complex is, in fact, a CW-complex. In order to do so we need the following result regarding the behavior of the trajectories of  $\varphi$ .

**Lemma 9.** *Let  $p_i$  and  $p_j$  be essential critical points of a Morse-Smale function  $f$  on a manifold with corners  $M$ . If  $p_j \in \overline{U(p_i)}$  then  $\text{index}(p_j) < \text{index}(p_i)$ . If  $p_i \in \overline{S(p_j)}$  then  $\text{index}(p_i) > \text{index}(p_j)$ .*

This lemma allows the maps  $\Phi_k$  and  $\Psi_k$  to be defined in a way that preserves  $\varphi$ -trajectories connecting critical points whose indices differ by one.

*Proof.* We will show that the boundaries of  $U(p_i)$  and of  $S(p_j)$  are composed of a union of trajectories. Suppose, in this case, that  $p_j \in \overline{U(p_i)}$ . Then there is a trajectory  $\tau \in S(p_j)$ , with  $\tau \subset \overline{U(p_i)}$ . The upper endpoint of  $\tau$  must also be a critical point  $p_k$  in  $\overline{U(p_i)}$ . Since  $f$  is a Morse-Smale function, it follows that  $\text{index}(p_k) > \text{index}(p_j)$ .

Traveling up from  $p_j$  in this way, there is a sequence of essential critical points with increasing indices. Since the topmost critical point in this sequence must be  $p_i$ , the result follows. The second half of the lemma is established in a similar way.

To show that the boundary of  $U(p_i)$  and of  $S(p_j)$  is composed of a union of trajectories, we will assume that  $q \in \overline{U(p_i)}$  and we will show two things. First, that for any  $t_0 > 0$ ,  $\varphi(t_0, q) \in \overline{U(p_i)}$  and second, that there exist  $t_0$  and  $q'$  such that  $q' \in \overline{U(p_i)}$  and  $\varphi(t_0, q') = q$ .

If  $q \in \overline{U(p_i)}$ , then there is a sequence  $\{q_n\} \subset U(p_i)$  such that  $\lim_{n \rightarrow \infty} q_n = q$ . Since  $\varphi(t_0, \cdot)$  is continuous, it follows that  $\lim_{n \rightarrow \infty} \varphi(t_0, q_n) = \varphi(t_0, q)$ .

The second assertion is more of a challenge, since  $\varphi$  provides a flow in only one direction. Again, let  $\{q_n\} \subset U(p_i)$  be a sequence such that  $\lim_{n \rightarrow \infty} q_n = q$ . Let  $r = f(q)$ .

If  $q$  is an essential critical point, then let  $q' = q$  and  $t_0$  be any real number greater than zero. Then  $\varphi(t_0, q') = q$ .

If  $q$  is not an essential critical point, let then for each  $n$ , let  $q'_n$  denote the intersection of the trajectory containing  $q_n$  with  $f^{-1}(r)$ . Then  $\lim_{n \rightarrow \infty} q'_n = q$  also.

Let  $r' > r = f(q)$  be such that  $f^{-1}([r, r'])$  contains no critical points (at least in a neighborhood of  $q$ ). Then let  $q''_n$  denote the intersection of the trajectory containing  $q_n$  with  $f^{-1}(r')$ . Since  $f^{-1}(r')$  is a closed submanifold, it is compact. There is a subsequence of  $\{q''_n\}$  that converges to a point  $q'$ . Relabel, if necessary, so that  $\lim_{n \rightarrow \infty} q''_n = q'$ . Note that  $q' \in \overline{U(p_i)}$ .

The map from  $f^{-1}(r')$  to  $f^{-1}(r)$  given by allowing each point of  $f^{-1}(r')$  to flow by  $\varphi$  until it hits  $f^{-1}(r)$  is both well defined and continuous (at least in a neighborhood of  $q'$ ). Since this map carries each  $q''_n$  to  $q'_n$ , it follows that the limit  $q'$  is carried to the corresponding limit  $q$ . Thus there is some  $t_0$  such that  $\varphi(t_0, q') = q$ .

It follows that  $\overline{U(p_i)}$  contains the entirety of some trajectory passing through  $q$ .

The above argument uses only the fact that  $\overline{U(p_i)}$  is a union of trajectories. The argument applies equally well to  $\overline{S(p_j)}$

q.e.d.

Note that this lemma does not imply that if  $\tau$  is a trajectory that intersects  $\overline{U(p_i)}$ , then  $\tau \subset \overline{U(p_i)}$ . Because trajectories combine wherever  $-\nabla f$  points outward, a point may lie in more than one trajectory. Of course, all trajectories containing a particular point must have the same lower limit.

**Lemma 10.** *The cellular complex  $X$  of the Morse-Smale function  $f : M \rightarrow \mathbb{R}$  is a CW-complex.*

*Proof.* By construction,  $\Sigma_i$  maps  $\partial P_i = \partial U_\varepsilon(p_i)$  onto a union of cells. The important point here is to show that these cells have dimension smaller than that of  $P_i$ .

Ideally, a cell  $P_j$  will be in the image of  $\Sigma_i$  if and only if there is a  $\varphi$ -trajectory connecting  $p_i$  to  $p_j$ .

By Lemma 9, for each critical point  $p_i$  there is an open set  $W$  containing  $p_i$ , such that  $U(p_j) \cap W = \emptyset$  whenever  $\text{index}(p_j) \leq \text{index}(p_i)$  and  $S(p_j) \cap W = \emptyset$  whenever  $\text{index}(p_j) \geq \text{index}(p_i)$ . We can then choose  $\varepsilon$  small enough that  $\Psi_{i+1}$  is the identity outside  $W$ .

Now suppose that  $S(p_j) \cap U(p_i) = \emptyset$ , but  $P_j \subset \text{image}(\Sigma_i)$ . The sets  $S(p_j)$  and  $U(p_i)$  must have been “confused” in a neighborhood of some essential critical point  $p_k$  by  $\Psi_{k+1}$ . It follows that  $\text{index}(p_k) < \text{index}(p_i)$  and  $\text{index}(p_j) < \text{index}(p_k)$ . It follows that

$$\text{index}(p_j) \leq \text{index}(p_i) - 2$$

Hence, the boundary of each cell is mapped to a union of lower dimensional cells, and the geometric complex is a CW-complex.

q.e.d.

The proof gives us some other information about the attaching maps of this CW-complex. Suppose that  $\Sigma$  is the attaching map for  $P = U_\varepsilon(p)$  and  $Q = U_\varepsilon(q)$  is a cell with dimension one less than that of  $P$ . Then  $\Sigma^{-1}(q)$  is exactly the set of points in  $\partial U_\varepsilon(q)$  that lie on trajectories connecting  $p$  to  $q$ .

This CW-complex gives rise to a chain complex

$$0 \longleftarrow W_0 \xleftarrow{\partial_1} W_1 \xleftarrow{\partial_2} \dots \xleftarrow{\partial_n} W_n.$$

The free Abelian groups  $W_i$  are generated by the  $i$ -cells of the above CW-complex. We will refer to this chain complex as the *geometric complex* of  $f : M \rightarrow \mathbb{R}$ .

**5.2. The Algebraic Complex of a Morse Function.** Let  $C_j(f)$  be the set of essential critical points of  $f$  with index  $j$ , and  $V_j$  the free Abelian group of formal  $\mathbb{Z}$ -linear combinations of the elements of  $C_j(f)$ . For each  $j = 1, \dots, n = \dim(M)$  define a map  $\partial'_j : V_j \rightarrow V_{j-1}$  by setting

$$\partial'_j(p) = \sum_{q \in C_{j-1}(f)} \text{degree}(p, q) q$$

for each  $p \in C_j(f)$ , and extending linearly to all of  $V_j$ . For  $j = 0$ , set  $\partial'_0(p) = 0$

In the next section, we will see that the free Abelian groups  $V_j$  and maps  $\partial'_j$  form a chain complex

$$0 \longleftarrow V_0 \xleftarrow{\partial'_1} V_1 \xleftarrow{\partial'_2} \dots \xleftarrow{\partial'_n} V_n$$

We will refer to this chain complex as the *algebraic complex* of  $f : M \rightarrow \mathbb{R}$ .

**5.3. Main Theorem.** We are now ready to state and prove the main result of this article.

**Theorem 11.** *Let  $M$  be a compact orientable  $n$ -manifold with corners, and  $f : M \rightarrow \mathbb{R}$  be a Morse-Smale function satisfying the conditions of Lemma 6. Then the algebraic complex of  $f : M \rightarrow \mathbb{R}$  is a chain complex. Moreover, the algebraic complex and geometric complex have the same homology groups.*

*Proof.* Let  $C_k(X)$  be the set of cells in  $X$  having dimension  $k$ . As above,  $C_k(f)$  is the set of critical points of  $f$  having index  $k$ . These sets are in one to one correspondence; each  $P \in C_k(X)$  satisfies  $P = U_\varepsilon(p)$  for some  $p \in C_k(f)$  and for each  $p \in C_k(f)$ ,  $P = U_\varepsilon(p) \in C_k(X)$ .

The bijection  $h_k : C_k(X) \rightarrow C_k(f)$  by  $U_\varepsilon(p) \mapsto p$  can be extended to an isomorphism  $h_k : W_k \rightarrow V_k$  by linearity. Thus we have a chain map  $h$  given by

$$\begin{array}{ccccccc} 0 & \longleftarrow & W_0 & \xleftarrow{\partial_1} & W_1 & \xleftarrow{\partial_2} & \dots & \xleftarrow{\partial_n} & W_n & \longleftarrow & 0 \\ \downarrow & & h_0 \downarrow & & h_1 \downarrow & & & & \downarrow & & h_n \downarrow \\ 0 & \longleftarrow & V_0 & \xleftarrow{\partial'_1} & V_1 & \xleftarrow{\partial'_2} & \dots & \xleftarrow{\partial'_n} & V_n & \longleftarrow & 0 \end{array}$$

Where each  $h_i$  is an isomorphism. It suffices to show that  $\partial_k = h_{k-1}^{-1} \circ \partial'_k \circ h_k$ .

The map  $\partial_k$  is defined by

$$\partial_k(P_j) = \sum_{P_i \in C_{k-1}(X)} \deg(\Sigma_j, P_i) P_i$$

for each  $P_j \in C_k(X)$ , where  $\Sigma_j$  is the attaching map for the cell  $P_j$ . The degree  $\deg(\Sigma_j, P_i)$  is determined by choosing a  $q \in P_i$  such that  $\Sigma_j(\partial P)$  intersects  $P_i$  transversely at  $q$ , and then adding the number of points in  $\Sigma_j^{-1}(q)$  at which  $\Sigma_j$  is orientation preserving, and subtracting the number of points in  $\Sigma_j^{-1}(q)$  at which  $\Sigma_j$  is orientation reversing. Given  $P_j \in C_k(X)$  and  $P_i \in C_{k-1}(X)$  we must determine this degree.

We have the relationship

$$\deg(\Sigma_j, P_i) = \deg(\Sigma_{ij}, U_\varepsilon(p_i)).$$

Consider a point  $q \in \partial U_\varepsilon(p_j) \cap \tau$  for some trajectory connecting  $p_j$  to  $p_i$ . The map  $\Sigma_{ij}$  is orientation preserving at  $q$  if  $\text{sign}(\tau) = +1$ , and orientation reversing if  $\text{sign}(\tau) = -1$ . It follows that

$$\deg(\Sigma_j, P_i) = \deg(\Sigma_{ij}, p_i) = \text{degree}(p_j, p_i).$$

and so

$$\deg(\Sigma_j, P_i) = \text{degree}(h_k(p_j), h_{k-1}(p_i)).$$

It follows that

$$\partial_k(P) = \sum_{P_i \in C_{k-1}(X)} \text{degree}(h_k(p_j), h_{k-1}(p_i)) h_{k-1}^{-1}(p_i).$$

and so  $\partial_k = h_{k-1}^{-1} \circ \partial'_k \circ h_k$  as claimed. This proves the theorem.

q.e.d.

It should be pointed out that in the the descending set of an essential critical point may not be homeomorphic to an open disk. Consequently, the descending sets may not be interpreted directly as cells in a CW-complex, as they may in the classical theory. Nonetheless, this theorem provides a useful parallel with the classical Morse theory in the setting of manifolds with corners.



## REFERENCES

- [Br] D. Braess. *Morse-Theorie für berandete Mannigfaltigkeiten*, Math. Ann., vol. 208, (1974), pp. 133-148.
- [GM] M. Goresky and R. MacPherson. *Stratified Morse Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete 14, Springer-Verlag, New York, 1988.
- [Ha1] H.A. Hamm. *Zum Homotopietyp Steinscher Räume*, J. Reine. Angew. Math., vol. 338 (1983), pp. 121-135.
- [Ha2] H.A. Hamm. *On Stratified Morse Theory*, Topology, vol. 38 (1999), pp. 427-438.
- [Han] D.G.C. Handron. *Generalized Billiard Paths and Morse Theory for Manifolds with Corners*, to appear in Topology and its Applications.
- [Mi] J. Milnor. *Morse Theory*. Annals of Mathematical Studies 51, Princeton University Press, Princeton, New Jersey, 1969.
- [Si] D. Siersma. *Singularities of Functions on Boundaries, Corners, Etc.*, Quart. J. Math. Oxford, vol. 2, (1981), pp. 119-127.
- [Va] S.A. Vakhrameev *Morse Lemmas for Smooth Functions on Manifolds With Corners* Journal of Mathematical Sciences, vol. 100 no. 4, (2000), pp. 2428-2445.

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA,  
(412)268-2545

*E-mail address:* `handron@andrew.cmu.edu`